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Deliverable D5.1

Report on classification of topological phase transitions in networks

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EXECUTIVE SUMMARY

The objective of the LINC project is to provide training and perform research in the development of network characterizations of climatic processes. In this context, the tasks in WP5 ``Tipping Points in the Climate System" include the development of network indicators of regime shifts that could be used to analyze states and predict evolution of the climate system close to tipping points. The first of the proposed tasks, which is the object of this deliverable, is a survey of the different transitions previously identified and characterized in networks, usually outside the framework of climate science.

This deliverable provides a concise listing of network transitions with the aim of giving LINC researchers an idea of the type of transitions one can expect in networks, what are the variables or tools used for their characterization, and to provide citations to the relevant literature. Transitions are classified depending on its nature of occurring *to the network* (as for example on percolation processes) or *on the network* (as for example when considering synchronization of oscillators in a network). A distinction is also made on whether they occur when varying a parameter (as in standard bifurcations) or rather they occur under time evolution.

Besides WP5, this deliverable is relevant for WP1 and WP4, since it applies to a variety of climatic networks and the concept of climate transitions and tipping points is tied to the concept of climate change and its prediction.

Deliverable Identification Sheet

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Abstract (for dissemination)	This deliverable provides a listing of transition types reported in networks, with some discussion on analysis tools and variables used to characterize them. Both transitions occurring on networks (such as synchronization) as transitions of the network topology itself (such as in percolation) are included here.	

Keywords	Tipping points, regime shifts, phase transitions, networks	

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Contents

1 Introduction

Regime transitions or shifts in the climate system are known to have occurred in the past. We show in Fig. 1 some examples surveyed by Dakos et al. (2008). They are abrupt changes in the state of the atmosphere and/or the ocean which impose important threads to biological organisms habiting the Earth. Methodologies to detect regime shifts in past time series have been developed, both in the climatic context and beyond. They usually consider the time evolution of local quantities (a proxy for temperature, and dust amount in sediments, for the examples in the figure) and search for statistical evidence of break points. Or instead researchers use classical indicators of temporal slowing down or enhanced fluctuations (Scheffer et al., 2012).

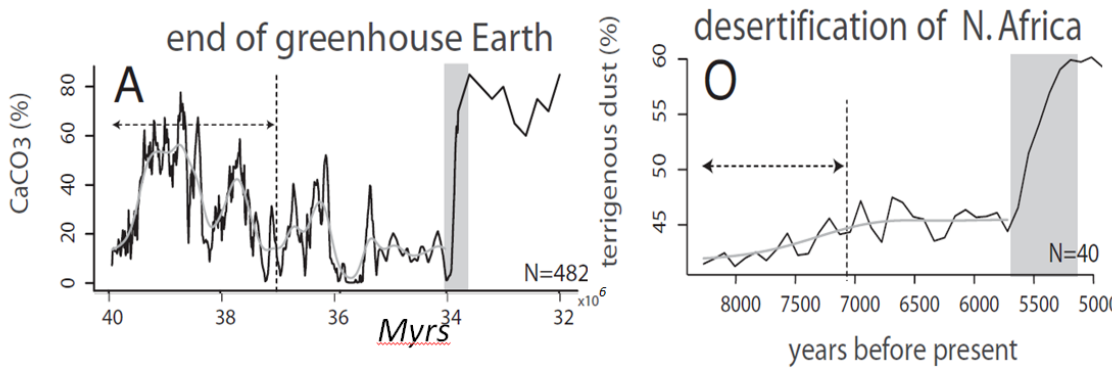

Figure 1: Two examples of abrupt transitions in paleoclimatic time series. Left: the end of the greenhouse state of the Earth occurred 34 millions of years ago, as shown by the abrupt increase of carbonate in sediments. Right: a more recent regime shift was the desertification of Northern Africa 5000–6000 years ago. Both plots from Dakos et al. (2008).

The core of the LINC project is the development of network characterizations of climatic processes that would take into account in a more holistic way available information. Instead of focusing on individual time series of temperature, precipitation, etc. the focus is shifted towards the *relationships* of these quantities among different places in the globe, the *climate networks*. We believe that the sensitivity of such approaches will be higher than of the classical ones. In this context, the tasks in WP5 *"Tipping Points in the Climate System"* include the development of network indicators of regime shifts that could be used to analyze states and predict evolution of the climate system. The first of the proposed tasks, which is the object of this deliverable, is a survey of the different transitions already identified in networks, usually in context different from climate research. The aim is to produce a concise listing that can give LINC researchers an idea of the type of transitions one can expect in networks, what are the variables or tools used for their characterization, and to provide pointers to the relevant literature.

In the following we describe a number of transition processes in networks. The number of instances of these phenomena reported in the literature is huge. Here we only mention those that can have some relevance in the study of climate networks, either because that kind of transition is likely to occur under climatic dynamics, or because this type of transition can occur in the dynamic variables that may be artificially introduced in the network to better elucidate its structure and topological properties (see Sect. 2).

For each type of transition we briefly define it and show the variables that have been used in its characterization. By no means we pretend to give a detailed description, describe the available theory, or summarize the main results. The only objective of this document is to give the reader an overview of the different transitions that may be encountered in the study of networks, so that they can be recognized if encountered in the study of climatic networks.

We finally mention here that most of the climatic networks considered so far are correlation networks. They are highly clustered and weighted. Also flow networks have bee built. They are, in addition to clustered and weighted, directed. These are quite difficult types of network and are never the first to be studied from the mathematical point of view when a new transition is discovered or introduced. This should be taken into account before comparing results available in the literature with the climate networks which are the subject of the LINC project.

2 Types of transition processes in networks

There is a large variety of different phenomena that could be considered in a way or another to be a *transition*. Although in the deliverable title the expression 'phase transition' is used, restriction to those will leave out many phenomena of interest in climate research. We consider a broader class of processes that can be of the following types:

The first mathematical concept that is associated to the idea of regime transition is that of *bifurcation*. A bifurcation is a qualitative change in the nature of the solutions of a dynamical system when a parameter is changed. *Tipping point* is the name given to the parameter value at which the relevant change occurs. The solutions involved in the change may be simple fixed points or more complicated regimes such as oscillations or chaotic dynamics. A *phase transition* is the technical name given to the analogous of bifurcations when occurring in infinitely large systems under noise. The framework of bifurcations is without doubt the more comfortable from the mathematical point of view since the different types of bifurcations have been carefully classified, and their consequences and warning signals thoroughly discussed. See Thompson and Sieber (2011) for a review relevant in the climate context.

But despite its mathematical convenience, standard bifurcations are not the objects commonly encountered when studying regime shifts in climate. One usually deals with observed time series, and the aim is to find indications of a qualitative change *in time*. There is a link between the two concepts: if there is a parameter in the system (such as an astronomical forcing, input of greenhouse gases into the atmosphere, melting water flowing from glaciers towards the sea, ...) changing very slowly in time then a qualitative change will occur in time when the changing parameter crosses the bifurcation or tipping time. But note that climate adaptation time scales should be faster than the rate of parameter change for this to occur cleanly. In many climatic processes this time-scale separation is not present, or occurs in the opposite direction.

There are cases however in which the bifurcation approach can be of use even if the climatic system under study is not undergoing it. It can be used as an analytical tool to investigate the structure of the network of interest. For example in climate research many networks are being build from correlations. Although these networks were not experiencing percolation processes (see sect. 3.1) in time, one may remove links with small values of correlation to identify the most correlated core, and continue the removal process until percolation as a tool to further explore the network structure.

On the other hand abrupt jumps unrelated to bifurcations can occur in dynamical systems. For example, in a bistable system forced by noise (as many models turn out to represent some climatic subsystems such as the Meridional Atlantic Circulation or the solar radiation - albedo feedback leading to snowball Earth states), jumps between the two stable states can occur stochastically because of the noise forcing, despite all parameters being kept fixed. This is an example of dynamic state which is composed by relatively long-lived subattractors, that the system can visit sequentially, giving rise to non-bifurcation-related regime shifts. Most instances of *community dynamics* discussed in Sect. 6 are probably of this type.

All the above applies to general dynamical systems, independently of being related to networks or not. When dealing specifically with climate networks we can distinguish two different types of transition phenomena: In the first one, the *topology* of the network changes (in time or by changing a parameter, as discussed above). In the second case, which happens when there are dynamical variables evolving at the nodes of the network (as for example temperature, pressure, etc. evolving dynamically at each node of a climate network), we can have transitions in the dynamics of these variables. These transitions will be greatly influenced by the nature of the variable's interactions, which are described by the network links, but they are not transitions of the network itself. Thus, we can talk of transitions *of the network*, and transitions *on the network* as two conceptually different cases. Finally we can have an hybrid case in which both the topology of the network and the dynamics on it experience simultaneous regime shifts. This happens in case of *coevolving* nodes and links . Also, as with the case of percolation mentioned above, one can consider a dynamic process running on top of a network, even if the process is unrelated to the climatic process represented in the network, if one wants to use it as a tool to explore network properties. For example one can define an unphysical oscillatory dynamics on a climate network since the synchronization properties and transitions of this system give information on the community structure and its transitions in the underling network.

In the rest of this document we will list and briefly describe different types of transitions reported in the literature. The next to chapters deal with transitions *of the network* and transitions of dynamical processes *on the network*. We treat in these sections both cases of transitions *in time* and transitions arising from *bifurcations*. Sects. 5 and **??** deal with situations in which both topology and dynamics on it become entangled.

3 Fragmentation-type transitions

3.1 Percolation

Percolation theory has a long history (Stauffer and Aharony, 1992) and it provides arguably the simplest example of a phase transition. It is a purely of geometric nature.

The basic concept is that of the *giant component*. It is defined in an infinite network as a set of mutually reachable vertices and their links, which happens to contain a finite fraction of the vertices of the infinite network. In more practical terms, one would have to identify the giant component by considering a finite network, computing the size of the largest connected component, and identifying if it diverges in size when the number of nodes grows. But this is not always feasible, specially if one deals with observed networks. In this case one identifies the largest component and decides whereas its size is of the order of the full network.

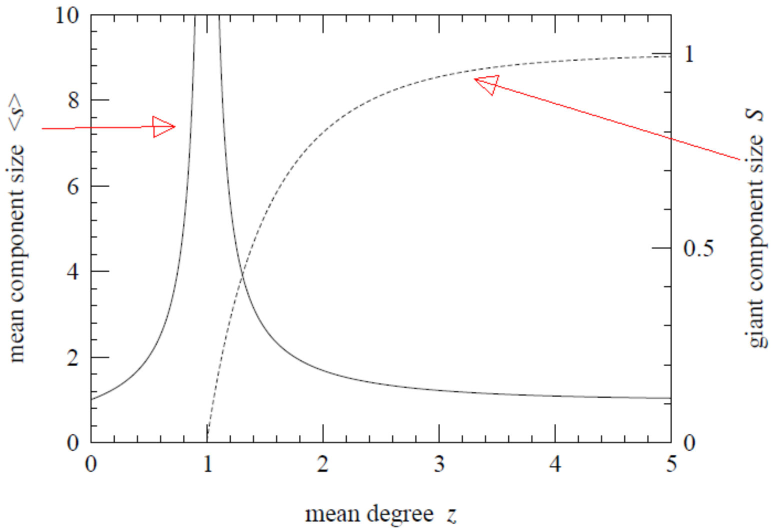

Figure 2: Percolation in a random Erdös-Rényi graph by varying the mean degree z. Solid line is the fraction of nodes in the giant component. Solid line is the mean cluster size excluding the giant component when it exists. From Newman (2003).

Figure 2 shows the behavior of the fraction S of nodes in the giant component for a random Erdös-Rényi network as a function of its mean degree z. It is seen that a giant component begins to exist when z is increased above $z = 1$. This is the percolation transition. The figure shows also that the distribution of the non-giant clusters also displays a singularity, a divergence, at the percolation point. The existence of a maximum in this second quantity, the mean component size excluding the giant, or also of a maximum in the size of the second largest cluster, are taken as identifiers of a percolation transition, and locators of the percolation point.

The Erdös-Rényi graph displays a percolation transition naturally as a function of its defining parameter. More in general one may try to induce a percolation transition in an arbitrary graph as follows: Let us start with a network containing a giant component, and eliminate randomly a fraction $f = 1 - p$ of the links (this is bond percolation; if one eliminate nodes, it is site percolation). One recalculates then the giant component and monitors it (or the size of the second cluster) as f increases or p decreases. A behavior such as the one in Fig. 2 with z replaced by p would delate the presence of a percolation transition.

3.2 Cascading processes

Here one considers also the situation of removing a fraction of the existing nodes in a network. In fact cascading processes in their simple form consider the removal of a single node (or link). But then, based in the new configuration of the network, a rule is given that leads to the removal of new nodes, and so on until the rule no longer request the elimination of further nodes and the cascade stops. In this case the order parameter G one monitors to identify a transition is the relative size of the giant component when the cascade stops with respect to the initial size (or equivalently, the size of the cascade or avalanche). The simplest rules for this cascading processes are inspired by congestion processes in communications or power systems: For example Motter and Lai (2002) considers each node to have a given capacity of communication, and also a load which is determined by its betweenness in the graph. After each event in the cascade loads are recalculated and the rule states that all nodes with load exceeding its capacity fail and are eliminated. Figure 3 shows the final relative size of the giant component, G, for different types of networks (and of removal process) as a function of a parameter α in their model that controls the assigned capacities: at $\alpha = 1$ all nodes are initially at their limiting load, whereas they are less stressed for increasing α.

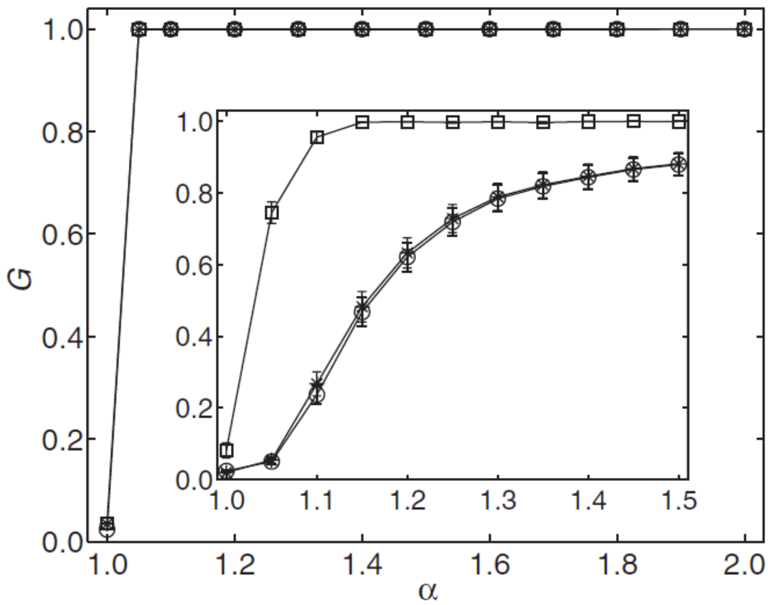

Figure 3: Relative size of the final largest connected component in the model of cascading failures of Motter and Lai (2002), as a function of a parameter α characterizing the initial tolerance of the nodes (ratio of the capacity to their initial load). Main panel, random graphs with degree $k = 3$. Inset, scale-free networks with $\langle k \rangle = 3$. Squares, circles and asterisks indicate that node removal is initiated at random, removing the node with largest load, or with the largest degree, respectively. From Motter and Lai (2002).

4 Dynamical processes on networks

4.1 Propagation and congestion

We have already seen in Section 3.2 examples of processes involving propagation and congestion, although formulated in purely geometric terms. If the model contains ingredients beyond topology, this should be taken into account when identifying the relevant order parameter to monitor when searching for transitions. We pose as example a model of Echenique et al. (2005). The process occurring in the network is packet passing, as in the Internet. The routing protocol by which a node decides to which neighbor to send a packet seeks to minimize the travel distance, but with a correction proportional to the congestion of the receiving node. In terms of the parameter h characterizing the importance of the distance criterium with respect to the congestion one, Echenique et al. (2005) computes and order parameter ρ which is the fraction of undelivered packets in the network at long times. The order parameter ρ reveals a jamming transition when the rate of packet input p increases. The transition is continuous for the standard internet protocol that takes into account only distance ($h = 1$) but it is discontinuous as soon as the congestion correction is introduced.

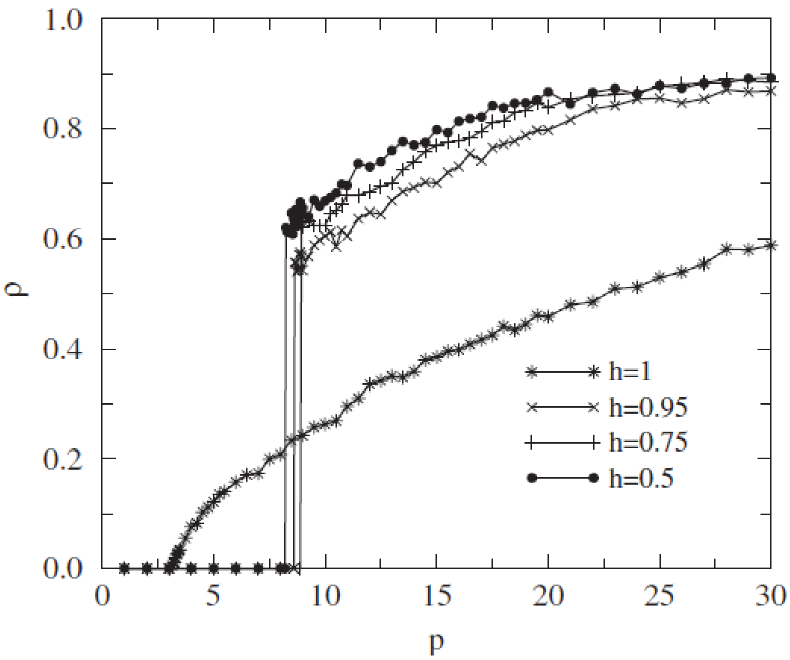

Figure 4: Order parameter ρ characterizing undelivered packets as a function of packet input rate p in the routing model of Echenique et al. (2005). $\rho > 0$ indicates a jammed phase, which appears discontinuously when node congestion is taken into account in the routing protocol ($h < 1$). From Echenique et al. (2005).

4.2 Epidemic spreading

Although the epidemic type of propagation is not expected to occur for the climatic variables in climatic networks, other types of propagation (waves, ...) may occur, and in addition the researcher may introduce such dynamics in some of the climate networks to better probe their structure. We will see that the insight one can get is similar to the one provided by percolation phenomena.

The most important of the basic models of epidemics are the so-called SIS and SIR models (Nâsell, 2002). S denotes susceptible to infection, I infective, and R recovered (or removed by death). Nodes in the networks are individuals which are in the state S, I or

R, and infections spread from vertex to vertex through edges. The SIS model describes infections without immunity, where recovered individuals are susceptible. In the SIR model, recovered individuals are immune forever, and do not infect.

The basic parameter in the epidemic models is the ratio λ between infection rate and recovery rate. The description of the different dynamic regimes is done in terms of the asymptotic *prevalence* which is the number of infected individuals at long times. When $\lambda < \lambda_c$ the prevalence is zero, meaning that infection dies out. In the SIR model also the asymptotic number of recovered individuals is zero. But when a epidemic threshold λ_c is exceeded, we have a nonzero prevalence (and growing with λ) in SIS and a nonzero number of recovered in the SIR model. The transition shares properties with percolation. More precisely the SIR model is equivalent (Grassberger, 1983) to directed percolation. Regarding the final state, it is nearly equivalent to a bond percolation problem in which $p = 1 - \exp(\lambda)$.

The analogy with percolation guarantees that for scale-free infinite networks the epidemic threshold is $\lambda_c = 0$: any infectivity leads to epidemic propagation (Pastor-Satorras and Vespignani, 2001).

5 Transitions associated to community structure

5.1 Spins and the Potts model

There is a large literature describing spin statistical mechanics models on networks. See a review in Dorogovtsev et al. (2008). Topology of the network defining the interactions greatly changes the macroscopic properties compared to the same model on a regular lattice. Here we only mention the spin models that turn out to be related to classical problems in graphs.

For example, the antiferromagnetic Ising model is defined as the statistical mechanics problem of a set of N spins with states $\{s1, ..., s_N\}$ with $s_i = \pm 1$, occupying the vertices of a network and interacting with energy

$$E = -J \sum A_{ij} s_i s_j \ .$$

(A_{ij}) is the adjacency matrix, the sum runs over all pairs of vertices, and $J < 0$. Finding the (zero temperature) ground state of this model turns out to be equivalent to the MAX-CUT problem of graph theory, namely finding the binary partition of a graph maximizing the number of links between the two parts. Each part is identified with the set of spins in the same state. Equivalently this is also the problem of coloring the vertices of a network with two colors so that no adjacent vertices have the same color. It turns out that the last problem may have or not solution depending of the network structure. The transition between such topologies is then equivalent to the transition between a fully antiferromagnetic state and a frustrated spin-glass state in the spin model. Studying the antiferromagnetic Ising model on a network is then a way to assess its degree of *bipartivity*.

An even richer statistical mechanics model that has been studied in networks is the q-state Potts model (Dorogovtsev et al., 2008). Here the spins $\{s_i\}$ can take q values, and the interaction energy is

$$E = -J \sum J_{ij} A_{ij} \delta_{s_i s_j} \ .$$

The sum is again over all pairs, (A_{ij}) is the graph adjacency matrix and the delta function is one or zero depending on whether their arguments are equal or different. There is a set of order parameters defined as

$$M_a = \sum_i \left(q \left\langle \delta_{s_i a} \right\rangle\right) / (q-1) \; .$$

All of them are zero in the paramagnetic phase, whereas some M_a are non zero if the corresponding phase with spins in state a becomes macroscopic. Phase transitions occur by varying the temperature, and depending on the value of q, of (J_{ij}) and of the network topology (A_{ij}), these transitions may be continuous or discontinuous.

The 2-state Potts model is clearly identical to the Ising model. The 1-state Potts model turns out to be equivalent to the problem of bond percolation. The Potts model has also remarkable relationships with graph-theoretic problems. For instance the antiferromagnetic case $J_{ij} = J < 0$ is related to the graph-coloring problem. Whereas planar graphs can always be colored with 4 colors in such a way that no contiguous nodes have the same color, the same is not generally true for more complex networks. Finding the ground state of the antiferromagnetic q-state Potts model gives the optimal coloring partition for q colors. Generally the perfect coloring is only achievable in networks with small-enough degree or large enough q, so that a q-COL-UNCOL transition between colorability and uncolorability occurs.

Perhaps the most remarkable relationship between Potts model and network theory is its application in community detection (Danon et al., 2005; Fortunato, 2010). Reichardt and Bornhold (2004) proposed to associate to a different community in a network each different domain in a q-state Potts model with a modified interaction energy given by

$$E = -\frac{1}{2} \sum_{\langle ij \rangle} A_{ij} \delta_{s_i s_j} + \frac{\lambda}{2} \sum_a n_a (n_a - 1) \; .$$

n_a is the number of spins in community a. Finding the ground state of such model partitions the network into q communities. In fact this modified energy when $\lambda = 1$ is exactly proportional to the network modularity, maximization of which is a standard method in community detection (Fortunato, 2010).

The relationship of the Potts model with different strategies to find communities in networks, and the presence of phase transitions in that model, implies that there are also phase transitions in community detection. As a simple example we show in Fig. 5 transitions in the efficiency of a method for finding communities in network, as the degree in the studied network changes (Nadakuditi and Newman, 2012).

5.2 Synchronization and the Laplacian matrix

Synchronization (in its simple instance of *complete synchronization*) is the adaptation of several interacting dynamic units to a common trajectory. We are here interested in the case in which the interactions are given by a network structure so that the dynamic units, or oscillators, are the nodes of a network. Many interesting questions and problems arise in this context. See reviews in Boccaletti et al. (2006); Dorogovtsev et al. (2008) we here only mention its relationship with community structure and detection.

An important observation was made by Arenas et al. (2006), who showed that the evolution at different time scales of synchronization patterns in a network with nested

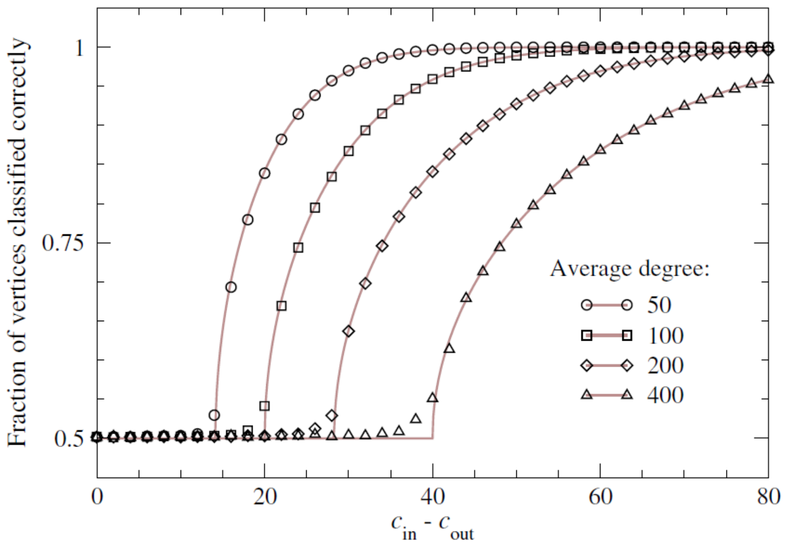

Figure 5: The number of nodes classified correctly by the spectral algorithm introduced by Nadakuditi and Newman (2012) when changing topological characteristics of the studied set of networks (namely some mean difference between in and out degree $d = c_{in} - c_{out}$). Phase transitions to increasing errors occurs when increasing d. From Nadakuditi and Newman (2012).

communities reveals its structure. Starting with a random initial conditions for a set of Kuramoto oscillators, highly interconnected clusters of nodes synchronize first. Then larger and larger communities merge into the common synchronized state until complete synchrony. These successive steps in synchronization can be understood as transitions occurring *in time*.

The explanation for this behavior can be found by analyzing the linear stability of the synchronized state. Linearization of the Kuramoto interaction leads to a diffusive coupling in which the *Laplacian* matrix of the graph mediates the interaction between nodes. For a network with adjacency matrix (A_ij) the Laplacian matrix is defined as $L_ij = k_i\delta_{ij} - A_{ij}$. k_i is the degree of node i. The Laplacian matrix appears also in the master stability approach to determine stability of synchronization of oscillators under standard diffusive-like coupling. This matrix has interesting properties and has been used in spectral methods of graph partitioning. In particular (L_{ij}) has an eigenvalue $\lambda_1 = 0$ associated to the uniform eigenvector (several zero eigenvalues associated to eigenvectors uniform in the different components occur for disconnected networks), and the rest of eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ are real and positive. It turns out (Arenas et al., 2006) that the different eigenvalues give the time-scale of synchronizations of the structures characterized by the corresponding eigenvalue. The eigenvalues of the Laplacian matrix also appear in the master stability estimation of synchronizability of networks, which becomes larger in standard cases when λ_N/λ_2 is smaller.

In fact the spectral properties of the Laplacian matrix were knew since long ago to provide information on the community structure of the corresponding network. It is in the basis of standard *graph partitioning algorithms* to solve the minimum cut problem: split the network in two components minimizing the number on links between them (Newman, 2010). An approximate solution of this problem is given by the eigenvalue associated to λ_2: positive elements identify the nodes in one of the components and negative ones in the other.

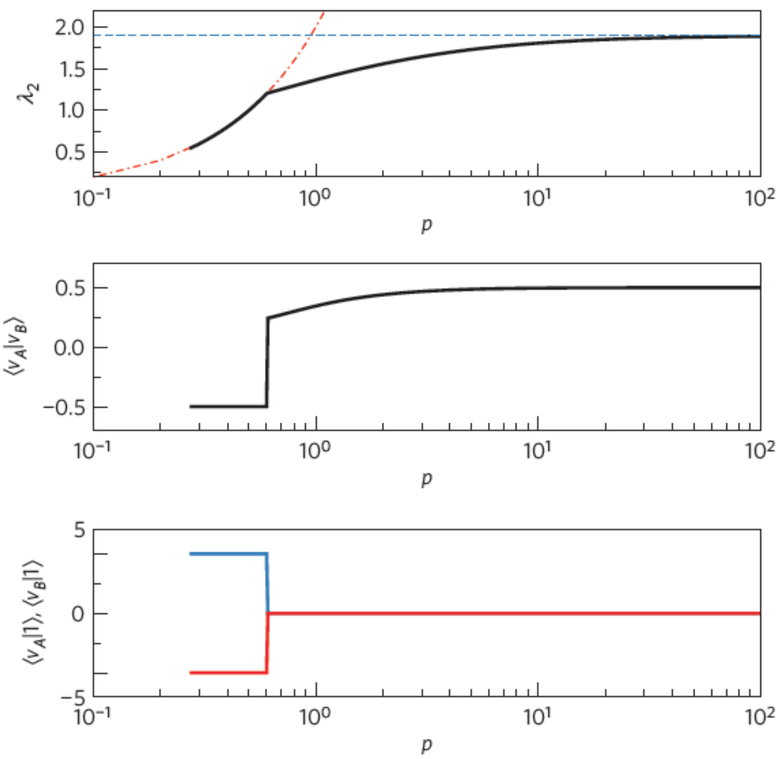

Figure 6: Topological transition occurring when increasing the weights of connections between two Erdös-Rényi networks with the same number of nodes. Top: the eigenvalue λ_2 of the total Laplacian matrix. Middle: scalar product of the parts of the corresponding eigenvalue that correspond to the first and to the second network. Bottom: The sum of the components of each of the two subeigenvectors. Increasing connectivity leads from a state in which the two networks are nearly independent to another in which the system behaves as a single network. From Arenas et al. (2006).

The spectral properties of the Laplacian have been used by Radicchi and Arenas (2013) to find and characterize a transition between a quasi-single layer and two-layer topology when increasing the number of links between two interconnected networks with the same number of nodes. Figure 6 displays the different indicators used to characterize the transition.

6 Community dynamics

We end this summary of transition types in networks by noticing that the elementary types of changes in community structure were enumerated by Palla et al. (2007). Figure 7 displays them.

Methodologies to identify and characterize these types of topological transitions are given for example in Aynaud et al. (2013), Mucha et al. (2010) and Peel and Clauset (2013).

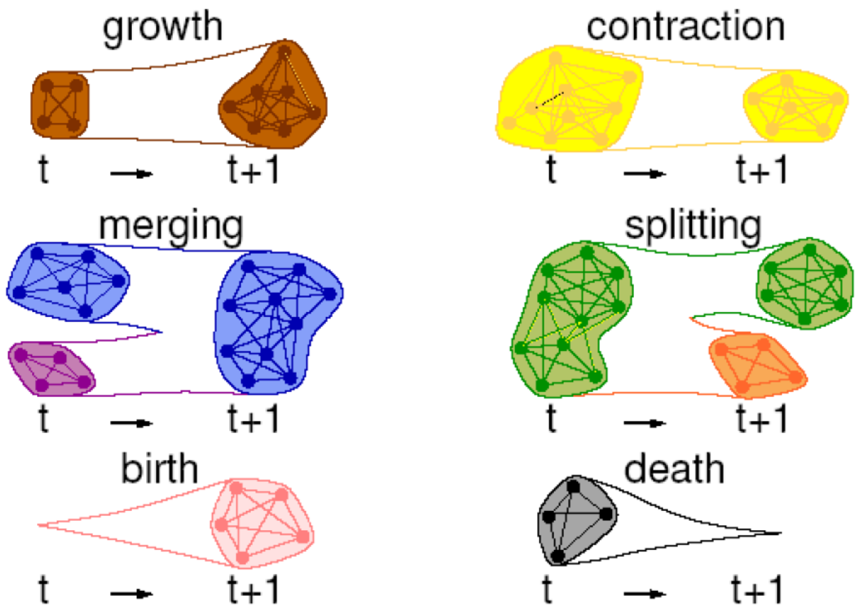

Figure 7: Elementary transitions in community dynamics in networks. From Palla et al. (2007).

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